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Small Displacements About Equilibrium of a Beam Subjected to Large Static Loads

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Small vibrations about the equilibrium configuration of a beam structure are treated. The beam is subjected to large static loads causing large displacements. The influence of these static loads on the small vibrations is investigated. In the stationary harmonic case, the symmetry of the dynamic stiffness matrix is discussed. A semifollower static moment is introduced to maintain the symmetry of the dynamic stiffness matrix and to create conservative subsystems. The beam is deformed only in bending and torsion, i.e., no extension of the geometric centerline is permitted and no transverse shear of the beam lamina is taken into account. The Euler–Bernoulli beam theory is applied to bending and the Saint-Venant theory to torsion. An Euler axis transformation is used for handling the large rotations in the nonlinear case. The material of the beam is assumed to be linearly elastic. A continuous model is made of the beam, i.e., the governing differential equations are derived and solved without recourse to a spatial discretization.

Nomenclature

$\boldsymbol{A}_X$, etc.	= matrix in governing differential equation, 1/m
$\boldsymbol{B}_X$, etc.	= matrix in governing differential equation, N/m, N
$\boldsymbol{E}$	= stiffness matrix, N/m, N, Nm
$\boldsymbol{EI}$	= elastic constitutive tensor, Nm^2
EI_η, EI_ζ	= bending stiffnesses about η and ζ, Nm^2
$\boldsymbol{e}$	= flexibility matrix, m/N, 1/N, 1/Nm
$\boldsymbol{e}_x$	= unit vector along beam in intermediate state, m
$\boldsymbol{e}_\xi$	= unit vector along beam in final state, m
$\boldsymbol{F}$	= cross-sectional force in beam, N
GI_t	= torsional stiffness about ξ, Nm^2
$\boldsymbol{I}$	= unit matrix, 1
$\boldsymbol{J}$	= unit symplectic matrix, 1
$\boldsymbol{L}_x$, $\boldsymbol{L}_\xi$	= rotation matrices from global to local coordinate system, 1
$\boldsymbol{M}$	= cross-sectional moment in beam, Nm
m	= mass per unit length, kg/m
$\boldsymbol{m}$	= mass tensor, kg/m
$\boldsymbol{P}$	= load array, N, Nm
$\boldsymbol{p}$	= displacement array, m, 1
$\boldsymbol{R}$	= position of geometric center of beam, m
s	= curvilinear coordinate along beam, m
$\boldsymbol{T}$	= transfer matrix, 1/m, 1, m, N/m, N, Nm, m/N, 1/N, 1/Nm
t	= time, s
$\boldsymbol{U}$	= externally applied force along beam, N/m
$\boldsymbol{u}$	= small translation of geometric center of beam in linear problem, m
$\boldsymbol{v}$	= large translation of geometric center of beam in nonlinear problem, m
$\boldsymbol{W}_X$	= matrix relating curvature and rotation, 1
XYZ	= global and inertial coordinate system, m
xyz, $\boldsymbol{x}$	= local coordinate system fixed to beam in intermediate state, m
$\boldsymbol{\kappa}$	= curvature (and twist) of beam, 1/m
$\xi\eta\zeta$, $\boldsymbol{\xi}$	= local coordinate system fixed to beam in final state, m
$\boldsymbol{\Phi}$	= externally applied moment along beam, N
$\boldsymbol{\phi}$	= small rotation of beam cross-section in linear problem, 1
$\boldsymbol{\varphi}$	= large Euler axis rotation of beam cross section in nonlinear problem, 1
ω	= angular frequency, 1/s

I. Introduction

THE aim of this study is to gain a better understanding of how to deal with small linear vibrations around the equilibrium configuration of a beam structure subjected to large static transverse and axial loads causing large displacements. The beam may bend and twist in the three-dimensional space. Beam kinematics of this type was studied at least as early as 1927 in the textbook by Love.[1] Other relatively early studies are reported by Reissner.[2,3] He determined the necessary kinematic expressions for the space-curved beam by use of the principle of virtual work.

Here the influence of large static loads on the small vibrations is investigated. When the beam undergoes stationary harmonic vibration, the symmetry of the dynamic stiffness matrix is checked under different handling schemes for the large static loads. The dynamic stiffness matrix relates the amplitudes of the small additional displacements to the small vectorially associated loads. A semifollower (semitangential) static moment is introduced to obtain a symmetric dynamic stiffness matrix and also to arrive at an infinitesimally symplectic set of differential equations (see Ref. 4). This semifollower moment can also be used for obtaining symmetric geometric stiffness matrices (see Ref. 5). The semifollower moment will result in conservative systems (see Refs. 4 and 6).

Here, the beam is deformed only in bending and torsion, i.e., no extension of the geometric centerline is permitted and no transverse shear of the beam lamina is taken into account. The Euler–Bernoulli beam theory is applied to bending and the Saint-Venant theory to torsion.

In a two-part paper, Minguet and Dugundji[7] studied similar problems for composite beams. However, in their second part they did not state how the large moment from the static analysis affects the linearized constitutive equations relating the small additional moments to the corresponding curvatures. In both Hughes[8] and Hodges,[9] different ways of obtaining rotation matrices for transforming from one coordinate system to another are given and also compared. The former applies the formulas to spinning rigid bodies, and the latter uses

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the equations in beam kinematics. The angular velocity of the rigid body corresponds to the curvature of the beam. This equivalence is part of Kirchhoff's kinetic analogy.

The Euler axis transformation used in the present study was employed by Argyris and Symeonidis[10] in nonlinear finite element analysis. Nonlinear beam kinematics was studied by Danielson and Hodges[11] and also by Simo and Vu-Quoc.[12] The former authors treated a beam undergoing large translations and rotations, but small strain. The latter authors studied beams undergoing large motions and apply a finite strain model. A recent study on nonlinear dynamics of three-dimensional beams was written by Simo et al.[13] Applications of the present study are foreseen to flexible space structures, i.e., spinning satellites outfitted with booms and antennas.

II. Governing Differential Equations

An initially straight beam is studied. A curvilinear coordinate s along the beam is introduced. The equilibrium equations in vectorial form for a beam lamina of length ds can be found in the literature, e.g., Wempner,[14] or can be easily determined:

$$\frac{\partial \boldsymbol{F}(s,t)}{\partial s} + \boldsymbol{U}(s,t) = \boldsymbol{0} \tag{1a}$$

$$\frac{\partial \boldsymbol{M}(s,t)}{\partial s} + \frac{\partial \boldsymbol{R}(s,t)}{\partial s} \times \boldsymbol{F}(s,t) + \boldsymbol{\Phi}(s,t) = \boldsymbol{0} \tag{1b}$$

Here, the vectors $\boldsymbol{F}(s,t)$ and $\boldsymbol{M}(s,t)$ are the sectional force and moment, respectively. The position vector of the geometric center of the beam cross section is denoted $\boldsymbol{R}(s,t)$. The d'Alembert forces and moments are included in the distributed external forces $\boldsymbol{U}(s,t)$ and moments $\boldsymbol{\Phi}(s,t)$ per unit length of beam. When the geometric centerline is inextensible, the unit vector along it is

$$\boldsymbol{e}_\xi(s,t) = \frac{\partial \boldsymbol{R}(s,t)}{\partial s} \tag{2}$$

Assuming that the material in the beam is linearly elastic, in the strain range of interest, the constitutive equations can be written in vector form as (the symbol $\cdot$ indicates contraction)

$$\boldsymbol{M}(s,t) = \boldsymbol{EI}(s)\cdot\boldsymbol{\kappa}(s,t) \tag{3}$$

Here, the parameter $\boldsymbol{EI}$ is the elastic constitutive tensor and the parameter $\boldsymbol{\kappa}(s,t)$ is the curvature vector. When Eq. (3) is projected onto the local coordinate system $\xi\eta\zeta$ with axes η and ζ along the principal axes of the beam cross section, the tensor $\boldsymbol{EI}$ is represented by a diagonal matrix with the torsional stiffness $GI_t(s)$ and the bending stiffnesses $EI_\eta(s)$ and $EI_\zeta(s)$ on the diagonal:

$$\boldsymbol{EI}_\xi(s) = \mathrm{diag}[GI_t(s) \quad EI_\eta(s) \quad EI_\zeta(s)] \tag{4}$$

Projecting Eq. (3) onto the axes of the principal frame $\xi\eta\zeta$ will give a relation between the components $\boldsymbol{M}_\xi(s,t)$ of the moment $\boldsymbol{M}(s,t)$ and the components $\boldsymbol{\kappa}_\xi(s,t)$ of the curvature $\boldsymbol{\kappa}(s,t)$ in the $\xi\eta\zeta$ frame,

$$\boldsymbol{M}_\xi(s,t) = \boldsymbol{EI}_\xi(s)\boldsymbol{\kappa}_\xi(s,t) \tag{5}$$

III. Static and Dynamic Cases

Instead of solving the equations in the preceding section directly, the aim is to superimpose a linear dynamic solution on a nonlinear static solution. As a special case the linear problem can also be static. The generalized sectional forces $\boldsymbol{M}(s,t)$ and $\boldsymbol{F}(s,t)$ and the externally applied loads $\boldsymbol{U}(s,t)$ and $\boldsymbol{\Phi}(s,t)$ along the beam are decomposed into one large static part and one small dynamic part,

$$\boldsymbol{F}(s,t) = \boldsymbol{F}_0(s) + \boldsymbol{F}_1(s,t) \tag{6a}$$

$$\boldsymbol{M}(s,t) = \boldsymbol{M}_0(s) + \boldsymbol{M}_1(s,t) \tag{6b}$$

$$\boldsymbol{U}(s,t) = \boldsymbol{U}_0(s) + \boldsymbol{U}_1(s,t) \tag{6c}$$

$$\boldsymbol{\Phi}(s,t) = \boldsymbol{\Phi}_0(s) + \boldsymbol{\Phi}_1(s,t) \tag{6d}$$

The position vector $\boldsymbol{R}(s,t)$ is decomposed into a static position and a dynamic displacement,

$$\boldsymbol{R}(s,t) = \boldsymbol{R}_0(s) + \boldsymbol{u}(s,t) \tag{7}$$

Here, the vector $\boldsymbol{R}_0(s)$ is the final position of the statically loaded beam including both the initial position and the displacement resulting from the static loads. The nonlinear static part of the equilibrium equations (1) is

$$\frac{\partial \boldsymbol{F}_0(s)}{\partial s} + \boldsymbol{U}_0(s) = \boldsymbol{0} \tag{8a}$$

$$\frac{\partial \boldsymbol{M}_0(s)}{\partial s} + \frac{\partial \boldsymbol{R}_0(s)}{\partial s} \times \boldsymbol{F}_0(s) + \boldsymbol{\Phi}_0(s) = \boldsymbol{0} \tag{8b}$$

The derivative of the position vector is again a unit vector along the geometric centerline of the beam, because no elongation of the centerline is permitted:

$$\frac{\partial \boldsymbol{R}_0(s)}{\partial s} = \boldsymbol{e}_x(s) \tag{9}$$

The linear dynamic parts of the equilibrium equations (1), when only the linear terms are kept, are

$$\frac{\partial \boldsymbol{F}_1(s,t)}{\partial s} + \boldsymbol{U}_1(s,t) = \boldsymbol{0} \tag{10a}$$

$$\frac{\partial \boldsymbol{M}_1(s,t)}{\partial s} + \frac{\partial \boldsymbol{R}_0(s)}{\partial s} \times \boldsymbol{F}_1(s,t) + \frac{\partial \boldsymbol{u}(s,t)}{\partial s} \times \boldsymbol{F}_0(s) + \boldsymbol{\Phi}_1(s,t) = \boldsymbol{0} \tag{10b}$$

The equilibrium equations (8) and (10) are written in vector form. They could also have been written in component form taking the components of the vectors in the global coordinate system, which would give a similar result. In Appendix A, Eqs. (8) and (10) are expressed in local components [see Eqs. (A1a), (A1b), (A2a), and (A2b)].

IV. Geometric and Kinematic Relations

Two local coordinate systems are introduced: the xyz system with the unit vector $\boldsymbol{e}_x(s)$ in the direction of the statically deformed beam axis of the intermediate state, and the $\xi\eta\zeta$ system with the unit vector $\boldsymbol{e}_\xi(s,t)$ along the deformed beam axis of the final (instantaneous) configuration. The rotation matrix $\boldsymbol{L}_x(s)$ relates components of vectors given in the local coordinate system xyz to their components in the global coordinate system XYZ:

$$\boldsymbol{x}(s) = \boldsymbol{L}_x(s)\boldsymbol{X} \tag{11}$$

The relationships between the components of curvature taken in local xyz and global XYZ coordinate systems can be expressed as

$$\hat{\boldsymbol{\kappa}}_{0x}(s) = \boldsymbol{L}_x(s)\frac{\partial \boldsymbol{L}_x^T(s)}{\partial s} \tag{12a}$$

$$\hat{\boldsymbol{\kappa}}_{0X}(s) = \boldsymbol{L}_x^T(s)\hat{\boldsymbol{\kappa}}_{0x}(s)\boldsymbol{L}_x(s) = \frac{\partial \boldsymbol{L}_x^T(s)}{\partial s}\boldsymbol{L}_x(s) \tag{12b}$$

e.g., see Hughes.[8] The matrix $\hat{\kappa}_{0x}$ is skew symmetric and built from the elements in the column matrix κ_{0x}; see Appendix C.

When there is no transverse shear of the beam lamina, the kinematic relation between the dynamic displacement vector $\boldsymbol{u}(s,t)$ of the geometric center of the beam cross section and the small rotation vector $\phi(s,t)$ of that cross section with respect to the static configuration is

$$\frac{\partial \boldsymbol{u}(s,t)}{\partial s} = \phi(s,t)\times\boldsymbol{e}_x(s) \tag{13}$$

Here, Eq. (13) is written in vector form, but one can also express it in local components in the xyz system; see Appendix A.

The components $\boldsymbol{\kappa}_\xi(s,t)$ of the curvature vector of the final configuration can be approximated linearly in the components $\phi_x(s,t)$ of the additional dynamic rotation of the cross section and in the components $\boldsymbol{\kappa}_{0x}(s,t)$ of the curvature vector of the statically deformed beam as

$$\boldsymbol{\kappa}_\xi(s,t) \approx \boldsymbol{\kappa}_{0\xi}(s,t) + \frac{\partial \phi_\xi(s,t)}{\partial s} \approx \boldsymbol{\kappa}_{0x}(s) - \hat{\phi}_x(s,t)\boldsymbol{\kappa}_{0x}(s) + \frac{\partial \phi_x(s,t)}{\partial s} \tag{14}$$

This is a relation between components of vectors taken in two different frames, i.e., the final frame $\xi\eta\zeta$ and intermediate frame xyz. Because the angles $\phi_x(s,t)$ or $\phi_\xi(s,t)$ between the two coordinate systems xyz and $\xi\eta\zeta$ are small, the linear transformation applied in Eq. (14) between the components in the two coordinate systems can be written

$$\boldsymbol{\xi}(s,t) \approx [\boldsymbol{I} - \hat{\boldsymbol{\phi}}_x(s,t)]\boldsymbol{x}(s) \tag{15}$$

V. Application of Constitutive Equations

The constitutive equation (5) is a relation between the components $\boldsymbol{M}_\xi(s,t)$ of moment $\boldsymbol{M}(s,t)$ taken in the local frame $\xi\eta\zeta$ and the components $\boldsymbol{\kappa}_\xi(s,t)$ of the curvature $\boldsymbol{\kappa}(s,t)$ in the same frame. The components $\boldsymbol{\kappa}_\xi(s,t)$ of the curvature are given in Eq. (14), and the corresponding equation for the components $\boldsymbol{M}_\xi(s,t)$ of the moment is

$$\begin{aligned}\boldsymbol{M}_\xi(s,t) &= \boldsymbol{M}_{0\xi}(s,t) + \boldsymbol{M}_{1\xi}(s,t) \\ &\approx \boldsymbol{M}_{0x}(s) - \hat{\boldsymbol{\phi}}_x(s,t)\boldsymbol{M}_{0x}(s) + \boldsymbol{M}_{1x}(s,t)\end{aligned} \tag{16}$$

Also here, the transformation (15) is used, because the angles $\phi_x(s,t)$ between the two coordinate systems xyz and $\xi\eta\zeta$ are small and only the linear terms have been retained in Eq. (16).

Combining Eqs. (5), (14), and (16) and separating the nonlinear static part from the linear dynamic part, one finds

$$\boldsymbol{M}_{0x}(s) = \boldsymbol{EI}_\xi(s)\boldsymbol{\kappa}_{0x}(s) \tag{17a}$$

$$\begin{aligned}\boldsymbol{M}_{1x}(s,t) &= \boldsymbol{EI}_\xi(s)\frac{\partial\boldsymbol{\phi}_x(s,t)}{\partial s} + \boldsymbol{EI}_\xi(s)\hat{\boldsymbol{\kappa}}_{0x}(s)\boldsymbol{\phi}_x(s,t) \\ &+ \hat{\boldsymbol{\phi}}_x(s,t)\boldsymbol{M}_{0x}(s)\end{aligned} \tag{17b}$$

Equations (17) can also be expressed as components in the global coordinate system:

$$\boldsymbol{M}_{0X}(s) = \boldsymbol{EI}_X(s)\boldsymbol{\kappa}_{0X}(s) \tag{18a}$$

$$\boldsymbol{M}_{1X}(s,t) = \boldsymbol{EI}_X(s)\frac{\partial\boldsymbol{\phi}_X(s,t)}{\partial s} + \hat{\boldsymbol{\phi}}_X(s,t)\boldsymbol{M}_{0X}(s) \tag{18b}$$

with the constitutive matrix

$$\boldsymbol{EI}_X(s) = \boldsymbol{L}_x^T(s)\boldsymbol{EI}_\xi(s)\boldsymbol{L}_x(s) \tag{18c}$$

and the derivative of the rotation vector projected in the global coordinate system XYZ:

$$\frac{\partial\boldsymbol{\phi}_X(s,t)}{\partial s} = \boldsymbol{L}_x^T(s)\left[\frac{\partial\boldsymbol{\phi}_x(s,t)}{\partial s} + \hat{\boldsymbol{\kappa}}_{0x}(s)\boldsymbol{\phi}_x(s,t)\right] \tag{18d}$$

VI. Infinitesimally Symplectic Form

It might be desirable to have a symmetric dynamic stiffness matrix for a beam element undergoing stationary harmonic vibration. A symmetric dynamic stiffness matrix corresponds to a symplectic transfer matrix; see Crellin and Janssens,[15] Müller,[16] and Appendix B. The transfer matrix can be calculated by integrating an infinitesimally symplectic system

$$\begin{aligned}&\frac{\partial}{\partial s}\left[\boldsymbol{u}_X^T \quad \boldsymbol{\phi}_X^T \quad \boldsymbol{F}_{1X}^T \quad \boldsymbol{M}_{\mathrm{is}X}^T\right]^T(s,t) \\ &= \boldsymbol{A}_{\mathrm{is}X}(s)\left[\boldsymbol{u}_X^T \quad \boldsymbol{\phi}_X^T \quad \boldsymbol{F}_{1X}^T \quad \boldsymbol{M}_{\mathrm{is}X}^T\right]^T(s,t) + \boldsymbol{B}_X(s,t)\end{aligned} \tag{19}$$

The matrix $\boldsymbol{A}_{\mathrm{is}X}(s)$ should be an infinitesimally symplectic matrix; again, see Appendix B. To be able to get the infinitesimally symplectic form, one has to introduce a semifollower static moment [consisting of the first two terms in Eq. (20) to follow] and also a new small dynamic moment $\boldsymbol{M}_{\mathrm{is}}(s,t)$ in Eq. (6b):

$$\boldsymbol{M}(s,t) = \boldsymbol{M}_0(s) + \tfrac{1}{2}\boldsymbol{\phi}(s,t)\times\boldsymbol{M}_0(s) + \boldsymbol{M}_{\mathrm{is}}(s,t) \tag{20}$$

The introduction of the semifollower moment is motivated by the fact that a structure loaded by a semifollower (semitangential) moment is a conservative system (see Ref. 4). A true moment can be replaced by at least three forces in one plane. A semifollower moment is created by letting these forces be directionally true, i.e., the forces keep their original directions when the system deforms. A system with directionally true forces is a conservative system (again, see Ref. 4). To obtain a conservative system, the externally applied distributed moment $\boldsymbol{\Phi}(s,t)$ in Eq. (6d) has to be chosen as a sum of a semifollower load and a small load $\boldsymbol{\Phi}_{\mathrm{is}}$:

$$\boldsymbol{\Phi}(s,t) = \boldsymbol{\Phi}_0(s) + \tfrac{1}{2}\boldsymbol{\phi}(s,t)\times\boldsymbol{\Phi}_0(s) + \boldsymbol{\Phi}_{\mathrm{is}}(s,t) \tag{21}$$

The vector equilibrium equation (10b) expressed with the moments $\boldsymbol{M}_{\mathrm{is}}(s,t)$ and $\boldsymbol{\Phi}_{\mathrm{is}}$ pertaining to the infinitesimally symplectic form can be written

$$\begin{aligned}&\frac{\partial\boldsymbol{M}_{\mathrm{is}}(s,t)}{\partial s} + \frac{1}{2}\frac{\partial\boldsymbol{\phi}(s,t)}{\partial s}\times\boldsymbol{M}_0(s) + \frac{1}{2}\boldsymbol{\phi}(s,t)\times\frac{\partial\boldsymbol{M}_0(s)}{\partial s} \\ &+ \frac{\partial\boldsymbol{R}_0(s)}{\partial s}\times\boldsymbol{F}_1(s,t) + \frac{\partial\boldsymbol{u}(s,t)}{\partial s}\times\boldsymbol{F}_0(s) + \frac{1}{2}\boldsymbol{\phi}(s,t) \\ &\times\boldsymbol{\Phi}_0(s) + \boldsymbol{\Phi}_{\mathrm{is}}(s,t) = \boldsymbol{0}\end{aligned} \tag{22}$$

Equation (22) is expressed in local coordinates in Appendix A. The constitutive equations (17b) and (18b) will also be modified when introducing the moment $\boldsymbol{M}_{\mathrm{is}}(s,t)$:

$$\begin{aligned}\boldsymbol{M}_{\mathrm{is}x}(s,t) &= \boldsymbol{EI}_\xi(s)\frac{\partial\boldsymbol{\phi}_x(s,t)}{\partial s} + \boldsymbol{EI}_\xi(s)\hat{\boldsymbol{\kappa}}_{0x}(s)\boldsymbol{\phi}_x(s,t) \\ &+ \frac{1}{2}\hat{\boldsymbol{\phi}}_x(s,t)\boldsymbol{M}_{0x}(s)\end{aligned} \tag{23}$$

$$\boldsymbol{M}_{\mathrm{is}X}(s,t) = \boldsymbol{EI}_X(s)\frac{\partial\boldsymbol{\phi}_X(s,t)}{\partial s} + \frac{1}{2}\hat{\boldsymbol{\phi}}_X(s,t)\boldsymbol{M}_{0X}(s) \tag{24}$$

The matrices in Eq. (19) can now be determined:

$$\boldsymbol{A}_{\mathrm{is}X}(s) = \begin{bmatrix}\boldsymbol{0} & -[\hat{\boldsymbol{e}}_x]_X & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \frac{1}{2}(\boldsymbol{EI}_X)^{-1}\hat{\boldsymbol{M}}_{0X} & \boldsymbol{0} & (\boldsymbol{EI}_X)^{-1} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_{\mathrm{is}X42} & -[\hat{\boldsymbol{e}}_x]_X & \frac{1}{2}\hat{\boldsymbol{M}}_{0X}(\boldsymbol{EI}_X)^{-1}\end{bmatrix} \tag{25a}$$

[see Eqs. (8b), (10a), (13), (22), and (24)] with

$$\boldsymbol{A}_{\mathrm{is}X42}(s) = \tfrac{1}{4}\hat{\boldsymbol{M}}_{0X}(\boldsymbol{EI}_X)^{-1}\hat{\boldsymbol{M}}_{0X} - \tfrac{1}{2}(\hat{\boldsymbol{F}}_{0X}[\hat{\boldsymbol{e}}_x]_X + [\hat{\boldsymbol{e}}_x]_X\hat{\boldsymbol{F}}_{0X}) \tag{25b}$$

and

$$\boldsymbol{B}_X(s,t) = \left[\boldsymbol{0}^T \quad \boldsymbol{0}^T \quad -\boldsymbol{U}_{1X}^T \quad -\boldsymbol{\Phi}_{\mathrm{is}X}^T\right]^T \tag{26}$$

A relation similar to Eq. (19) can be determined without attempting to get an infinitesimally symplectic set of differential equations and without introducing the moment $\boldsymbol{M}_{\mathrm{is}}(s,t)$. This new relation contains the same parameters as Eq. (19) with the exception that the moment $\boldsymbol{M}_{\mathrm{is}X}(s,t)$ is replaced by the moment $\boldsymbol{M}_{1X}(s,t)$. The semifollower external distributed moment is still used, because the same physical system is considered. The matrix $\boldsymbol{A}_X(s)$ corresponding to the matrix $\boldsymbol{A}_{\mathrm{is}X}(s)$ in Eqs. (25) for the case when no semifollower cross-sectional moment is introduced is

$$\boldsymbol{A}_X(s) = \begin{bmatrix}\boldsymbol{0} & -[\hat{\boldsymbol{e}}_x]_X & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & (\boldsymbol{EI}_X)^{-1}\hat{\boldsymbol{M}}_{0X} & \boldsymbol{0} & (\boldsymbol{EI}_X)^{-1} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \frac{1}{2}\hat{\boldsymbol{\Phi}}_{0X} - \hat{\boldsymbol{F}}_{0X}[\hat{\boldsymbol{e}}_x]_X & -[\hat{\boldsymbol{e}}_x]_X & \boldsymbol{0}\end{bmatrix} \tag{27}$$

[see Eqs. (10), (13), and (18b)]. It is also possible to calculate a dynamic stiffness matrix for the beam element from the differential equation with the governing matrix $\boldsymbol{A}_X(s)$. However, if there is a static moment applied to at least one of the two boundaries, this matrix will be nonsymmetric unless this static moment is semifollowing the rotation of the beam end. Thus, if a dynamic stiffness matrix relates translations u_X and rotations ϕ_X at the ends to the forces F_X and moments M_{isX} defined in Eq. (20), it will be symmetric. The corresponding equations for the local coordinates to Eqs. (19) and (25–27) are given in Appendix A.

VII. Nonlinear Differential Equations

Instead of superimposing two solutions of a nonlinear static problem and a linear dynamic or static problem, one can solve the complete nonlinear problem. This is easily accomplished when both cases are static. A few kinematic relations are missing both in this complete problem and in the static nonlinear problem of Sec. III as consisting of Eqs. (5), (8), and (9). An Euler axis transformation is adopted (e.g., see Ref. 8 or 6). The transformation between components in the local coordinate system $\xi\eta\zeta$ and the global coordinate system XYZ can be written

$$\xi(s,t) = L_\xi(s,t)X \tag{28a}$$

with

$$L_\xi(s,t) = I - \frac{\sin|\varphi_X|}{|\varphi_X|}\hat{\varphi}_X + \frac{1-\cos|\varphi_X|}{|\varphi_X|^2}\hat{\varphi}_X\hat{\varphi}_X \tag{28b}$$

and

$$|\varphi_X(s,t)| = \sqrt{\varphi_X^T(s,t)\varphi_X(s,t)} \tag{28c}$$

The expression for the curvature can be obtained from Eqs. (12b) and (28b) as

$$\kappa_X(s,t) = W_X(s,t)\frac{\partial\varphi_X(s,t)}{\partial s} \tag{29a}$$

with

$$W_X(s,t) = I + \frac{1-\cos|\varphi_X|}{|\varphi_X|^2}\hat{\varphi}_X + \frac{|\varphi_X| - \sin|\varphi_X|}{|\varphi_X|^3}\hat{\varphi}_X\hat{\varphi}_X \tag{29b}$$

The nonlinear problem consisting of Eqs. (1), (2), (5), (28a), and (29a) can be rearranged in a systematical way expressed in global components as

$$\frac{\partial R_X(s,t)}{\partial s} = L_\xi^T(s,t)[e_\xi]_\xi \tag{30a}$$

$$\frac{\partial\varphi_X(s,t)}{\partial s} = W_X^{-1}(s,t)L_\xi^T(s,t)(EI_\xi)^{-1}(s)L_\xi(s,t)M_X(s,t) \tag{30b}$$

$$\frac{\partial F_X(s,t)}{\partial s} = -U_X(s,t) \tag{30c}$$

$$\frac{\partial M_X(s,t)}{\partial s} = \hat{F}_X(s,t)L_\xi^T(s,t)[e_\xi]_\xi - \Phi_X(s,t) \tag{30d}$$

VIII. Examples

In the following examples, a uniform Euler–Bernoulli cantilever beam is treated. In examples B and C, both the nonlinear and the linear cases are static. The linear case in example D is stationary harmonic.

Example A: Nonlinear Statically Loaded Cantilever Beam

An initially straight uniform cantilever beam subjected to a large static transverse force P_{0Z} applied at the kinematically free end is studied (Fig. 1). The beam will deform in one plane and the governing nonlinear differential equations can be written

$$\frac{dR_{0X}(s)}{ds} = \cos\varphi_{0Y}(s) \tag{31a}$$

$$\frac{dR_{0Z}(s)}{ds} = -\sin\varphi_{0Y}(s) \tag{31b}$$

$$\frac{d\varphi_{0Y}(s)}{ds} = \frac{M_{0Y}(s)}{EI_\eta} \tag{31c}$$

$$\frac{dF_{0Z}(s)}{ds} = -U_{0Z}(s) \tag{31d}$$

$$\frac{dM_{0Y}(s)}{ds} = F_{0Z}(s)\cos\varphi_{0Y}(s) - \Phi_{0Y}(s) \tag{31e}$$

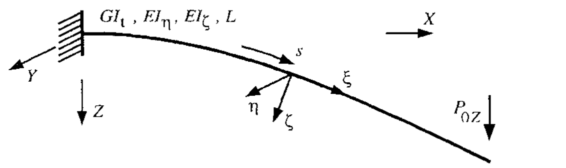

Fig. 1 Perspective view of initially straight uniform beam statically displaced and deformed by directionally true transverse load P_{0Z} at kinematically free end; beam has length L, bending stiffnesses EI_η and $EI_\zeta = 2EI_\eta$, and torsional stiffness $GI_t = EI_\eta$; only plane bending occurs here.

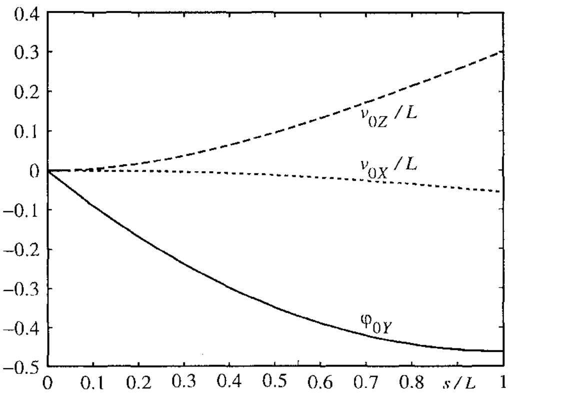

Fig. 2 Static translations $v_{0X}(s)$ and $v_{0Z}(s)$ of geometric centerline of beam and rotation $\varphi_{0Y}(s)$ of beam cross section plotted dimensionlessly vs curvilinear coordinate s along beam; load parameter $P_{0Z}L^2/EI_\eta$ is equal to unity.

Here, the distributed external loads $U_{0Z}(s)$ and $\Phi_{0Y}(s)$ are zero. The boundary conditions are the position $R_0(s = 0) = \mathbf{0}$, the rotation $\varphi_{0Y}(s = 0) = 0$, the force $F_{0Z}(s = L) = P_{0Z}$, and the moment $M_{0Y}(s = L) = 0$. The nonlinear differential equations (31) can be obtained by simplifying Eqs. (30) for the nonlinear three-dimensional case. However, the basic equations for a beam undergoing large deflection in one plane can be found, e.g., in Timoshenko and Gere,[17] where postbuckling of beams is treated.

The differential equations are solved iteratively. In each iteration the differential equations are integrated from the clamped end to the free end using a fourth-order Runge–Kutta scheme dividing the beam into 1000 elements. This extreme choice of the number of elements was made because an accurate solution was sought. Also in example B, a high resolution of the static bending moment as a function of the curvilinear coordinate will be needed. Before each iteration, a new bending moment at the clamped end has to be calculated. It is obtained by multiplying the applied load at the free end by the lever arm between the clamped end and the free end, which is given by the position of the free end as calculated in the previous iteration. The number of iterations needed was 18 until the absolute value of the boundary moment $M_{0Y}(s = L)$ at the free end was less than $10^{-12}EI_\eta/L$.

In Fig. 1 the calculated equilibrium configuration of the deformed beam is shown, when the load parameter $P_{0Z}L^2/EI_\eta$ is taken as unity. In Fig. 2 the large translations $v_{0X}(s) = R_{0X}(s) - s$ and $v_{0Z}(s) = R_{0Z}(s)$ of the geometric center of the beam and the large rotation $\varphi_{0Y}(s)$ of the beam cross section are plotted dimensionlessly as functions of the dimensionless curvilinear coordinate s/L. The solution achieved has at least 12 correct digits. The calculated maximum vertical translation $v_{0Z}(s = L)$ is 30.17% of the beam length.

Example B: Linear Flexibility at Tip of Beam

In this example the static flexibility matrix of the tip of the beam is calculated around the equilibrium configuration determined in example A. The tip is defined as the right-hand end of the beam ($s = L$), which was kinematically free in example A. Here, a dimensionless flexibility matrix e is defined by a relation between the small dimensionless displacements $\boldsymbol{p}(L)$ and the vectorially associated small dimensionless loads $\boldsymbol{P}(L)$ at the tip of the beam,

$$\boldsymbol{p}(L) = \boldsymbol{e}\boldsymbol{P}(L) \tag{32a}$$

with

$$\boldsymbol{p}(L) = \begin{bmatrix} \frac{u_X(L)}{L} & \frac{u_Y(L)}{L} & \frac{u_Z(L)}{L} & \phi_X(L) & \phi_Y(L) & \phi_Z(L) \end{bmatrix}^T \tag{32b}$$

$$\boldsymbol{P}(L) = (L/EI_\eta) \times \left[F_X(L)L \;\; F_Y(L)L \;\; F_Z(L)L \;\; M_X(L) \;\; M_Y(L) \;\; M_Z(L)\right]^T \tag{32c}$$

Each column of the flexibility matrix is the additional displacements in the six degrees of freedom, when an additional unit load associated with one degree of freedom is applied to the beam and when the beam remains unloaded in the other degrees of freedom with the exception of the large static load P_{0Z} that is always present. The flexibility matrix at the tip will be symmetric independently of the choice between a semifollower static moment and a nonfollower static moment, because the static moment at the tip is zero. When the numbers are truncated, the flexibility matrix at the tip is

$$\boldsymbol{e} = \begin{bmatrix} 0.037840 & 0.000000 & -0.096278 & 0.000000 & 0.161305 & 0.000000 \\ & 0.157675 & 0.000000 & -0.087545 & 0.000000 & 0.218524 \\ & & 0.248876 & 0.000000 & -0.395683 & 0.000000 \\ & & & 0.884795 & 0.000000 & 0.204396 \\ & & & & 0.883761 & 0.000000 \\ S & Y & M & & & 0.576596 \end{bmatrix} \tag{33}$$

The minus sign in Eq. (35) depends on the fact that the boundary generalized forces act in opposite direction to the applied loads vectorially associated with the displacements at the left-hand end of the beam. The flexibility matrix $\boldsymbol{e}$ at the tip can be calculated as

$$\boldsymbol{e} = \boldsymbol{T}_{12}\boldsymbol{T}_{22}^{-1} \tag{36}$$

The number of correct digits in Eq. (33) is assumed to be the same as for the previous nonlinear solution in example A.

The flexibility matrix was also calculated approximately by solving the differential equations (30). First, the equations were solved only with the large transverse force from example A, and the results were the same as before. Second, six nonlinear loading cases were solved with the large transverse load at the tip plus a small load at the tip in one of the six directions. Dimensionless forces and moments are used and the small load is taken as 1/1000 of the large load. The linear flexibility matrix is estimated with the difference, multiplied by 1000, between the displacements of the tip for these six loading cases and the loading case with only the large transverse force. However, large and small rotations are not additive, which means that the small rotation $\phi_X(s = L)$ of the tip has to be calculated from the rotation matrix $\boldsymbol{L}_{\xi x}[\boldsymbol{\phi}_x(L)]$ defined by

$$\begin{aligned}\boldsymbol{\xi}(L) &= \boldsymbol{L}_\xi[\boldsymbol{\varphi}_X(L)]\boldsymbol{X} = \boldsymbol{L}_\xi[\boldsymbol{\varphi}_X(L)]\boldsymbol{L}_x^T[\boldsymbol{\varphi}_{0X}(L)]\boldsymbol{x}(L) \\ &= \boldsymbol{L}_{\xi x}[\boldsymbol{\phi}_x(L)]\boldsymbol{x}(L)\end{aligned} \tag{37}$$

A standard fourth- and fifth-order Runge–Kutta scheme was used within MATLAB™ (Ref. 18). Here, the differential equations (19) have been solved for six sets of boundary conditions,

$$\begin{bmatrix} \boldsymbol{u}_X(0) \\ \boldsymbol{\phi}_X(0) \end{bmatrix} = \boldsymbol{0} \tag{34a}$$

$$\begin{bmatrix} \boldsymbol{F}_{1X}(0) \\ \boldsymbol{M}_{\mathrm{is}\,X}(0) \end{bmatrix} = \boldsymbol{I} \tag{34b}$$

The set of boundary conditions (34) will result in two blocks of the transfer matrix $\boldsymbol{T}$. The transfer matrix is the relation between the displacements and loads at the beam end $s = L$ and the same variables at $s = 0$:

$$\begin{bmatrix} \boldsymbol{p}(L) \\ \boldsymbol{P}(L) \end{bmatrix} = \begin{bmatrix} \boldsymbol{T}_{11} & \boldsymbol{T}_{12} \\ \boldsymbol{T}_{21} & \boldsymbol{T}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{p}(0) \\ -\boldsymbol{P}(0) \end{bmatrix} \tag{35}$$

[see Eqs. (11), (28a), and (28b)]. The Euler vector $\boldsymbol{\varphi}_{0X}(L)$ is the rotation from the global coordinate system XYZ to the local coordinate system $\boldsymbol{x}(L)$ at the tip of the beam, when it is subjected only to the large transverse force P_{0Z}. The rotation $\boldsymbol{\varphi}_X(L)$ and the local coordinate system $\boldsymbol{\xi}(L)$ are defined in the same manner for each one of the six loading cases already discussed. The rotation $\boldsymbol{\phi}_X(L)$ of interest is the Euler vector for rotating from the $\boldsymbol{x}(L)$ frame to the $\boldsymbol{\xi}(L)$ frame taken in the global coordinate system XYZ:

$$\boldsymbol{\phi}_X(s = L) = \boldsymbol{L}_x^T[\boldsymbol{\varphi}_{0X}(L)]\boldsymbol{\phi}_x(L) \tag{38}$$

The flexibility matrix calculated becomes

$$\boldsymbol{e} = \begin{bmatrix} 0.037823 & -0.000013 & -0.096299 & -0.000001 & 0.161208 & -0.000026 \\ 0.000000 & 0.157675 & 0.000000 & -0.087545 & 0.000000 & 0.218524 \\ -0.096252 & -0.000008 & 0.248814 & -0.000013 & -0.395773 & -0.000019 \\ 0.000000 & -0.087545 & 0.000000 & 0.884794 & 0.000000 & 0.204396 \\ 0.161256 & 0.000004 & -0.395603 & -0.000065 & 0.883835 & 0.000104 \\ 0.000000 & 0.218524 & 0.000000 & 0.204396 & 0.000000 & 0.576596 \end{bmatrix} \tag{39}$$

The flexibility matrix in Eq. (39) should not be perfectly symmetric. However, the nonsymmetry will diminish when the small additional loads are made even smaller. Here, a better result can also be achieved by subtracting two sets of loading cases: one set will be a set of small positive loads and the other a set of small negative loads. In both sets the large load will be included.

Example C: Linear Stiffness at Root of Beam

In this example the static stiffness matrix at the root is calculated around the equilibrium configuration calculated in example A, when the beam is released from the wall. The root is defined as the left-hand end of the beam ($s = 0$), which was clamped in example A. Only the rotational degrees of freedom and the associated moments are considered. Here, each column of the stiffness matrix is the small additional moments in the chosen degrees of freedom that have to be applied at the root, when the root undergoes a unit rotation around one of the coordinate axes X, Y, or Z. The large static load P_{0Z} is always acting. Because the static moment at the root is nonzero, the stiffness matrix can be symmetric or nonsymmetric, depending on which small moment $\boldsymbol{M}_{\text{is}}$ or $\boldsymbol{M}_1$ is related to the rotations at the root. The following results were obtained:

$$-\frac{L}{EI_\eta}\begin{bmatrix} M_{\text{is}\,X}(0) \\ M_{\text{is}\,Y}(0) \\ M_{\text{is}\,Z}(0) \end{bmatrix} = \begin{bmatrix} 0.144045 & 0.000000 & -0.471783 \\ 0.000000 & 0.263881 & 0.000000 \\ -0.471783 & 0.000000 & 0.000000 \end{bmatrix}\begin{bmatrix} \phi_X(0) \\ \phi_Y(0) \\ \phi_Z(0) \end{bmatrix} \tag{40a}$$

$$-\frac{L}{EI_\eta}\begin{bmatrix} M_{1X}(0) \\ M_{1Y}(0) \\ M_{1Z}(0) \end{bmatrix} = \begin{bmatrix} 0.144045 & 0.000000 & -0.943567 \\ 0.000000 & 0.263881 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 \end{bmatrix}\begin{bmatrix} \phi_X(0) \\ \phi_Y(0) \\ \phi_Z(0) \end{bmatrix} \tag{40b}$$

The calculations were similar to the calculations in the first part of example B. The transfer matrix $\boldsymbol{T}$ of Eq. (35) can be calculated by integrating the differential equation (19) with $[\boldsymbol{p}^{\mathrm{T}}(0) \quad -\boldsymbol{P}^T(0)]^T$ in the boundary conditions at the left-hand end taken as the 12×12 unit matrix $\boldsymbol{I}$. When the tip still can move freely with the large static load P_{0Z}, the stiffness matrix at the root is

$$\boldsymbol{E} = \boldsymbol{T}_{22}^{-1}\boldsymbol{T}_{21} \tag{41}$$

The matrices in Eqs. (40) are the last three rows and columns of the stiffness matrix in Eq. (41) calculated with a semifollower large static moment or a nonfollower static moment. The difference between these two matrices in Eqs. (40) is equal to the matrix, when the load parameter $P_{0Z}L^2/EI_\eta$ is equal to unity:

$$\frac{-\hat{M}_{0X}(s=0)}{2} = P_{0Z}L\begin{bmatrix} 0.000000 & 0.000000 & 0.471783 \\ 0.000000 & 0.000000 & 0.000000 \\ -0.471783 & 0.000000 & 0.000000 \end{bmatrix} \tag{42}$$

The confusing minus signs in the left-hand parts of Eqs. (40) arise from the fact that the boundary moments act in opposite direction to the applied moments associated with the rotations $\phi_X(s = 0)$ at the root.

Example D: Harmonically Loaded Beam

Forced harmonic undamped vibration about the equilibrium configuration of the beam in Fig. 1 will be studied. Only translatory inertia is taken into account, i.e., the distributed loads in Eqs. (26) and (A7) will be given by

$$\boldsymbol{U}(s,t) = -\boldsymbol{m}(s)\cdot\ddot{\boldsymbol{u}}(s,t) \tag{43a}$$

$$\boldsymbol{\Phi}(s,t) = \boldsymbol{0} \tag{43b}$$

Here, $\boldsymbol{m}(s)$ is the mass tensor, which projected in the local coordinate system $\xi\eta\zeta$ is represented by the diagonal matrix

$$\boldsymbol{m}_\xi(s) = m(s)\boldsymbol{I} \tag{44}$$

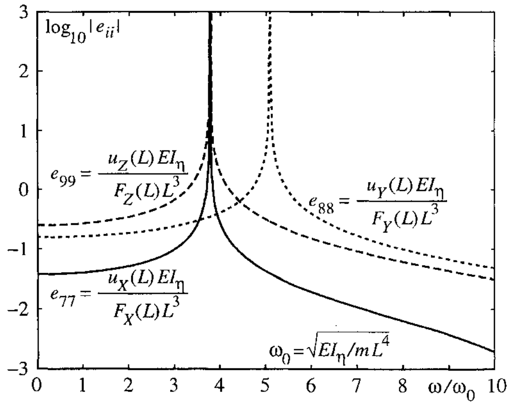

Fig. 3 Calculated dimensionless dynamic direct flexibilities e_{77}, e_{88}, and e_{99}, pertaining to translations $u_X(L)$, $u_Y(L)$, and $u_Z(L)$ at the tip plotted vs dimensionless frequency $\omega\sqrt{(mL^4/EI_\eta)}$.

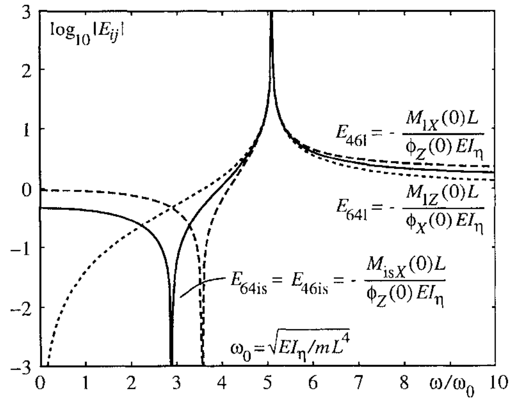

Fig. 4 Calculated dimensionless cross-stiffness elements E_{46} and E_{64}, pertaining to rotation of the root about global axes X and Z, plotted vs dimensionless frequency $\omega\sqrt{(mL^4/EI_\eta)}$ for both a nonfollower and a semifollower large static moment, which acts at the root.

Here, only the translatory inertia of the beam itself is considered, i.e., no ambient mass is excited giving any contribution to the translatory inertia. Elimination of the time dependence by introducing a harmonic time function will give the amplitude of the distributed load taken in the global coordinate system as

$$U_{1X}(s) = m(s)\omega^2\boldsymbol{u}_X(s) \tag{45}$$

The mass per unit length $m(s) = m$ is taken as constant along the beam.

In Fig. 3 the dimensionless direct flexibilities pertaining to the translations $u_X(L)$, $u_Y(L)$, and $u_Z(L)$ of the tip are plotted vs the dimensionless frequency $\omega\sqrt{(mL^4/EI_\eta)}$. The flexibilities are defined as in Eqs. (32) and calculated as in Eq. (36). The first three diagonal elements in Eq. (33) are found for zero frequency in Fig. 3. The peaks at the dimensionless frequencies, 3.771 and 5.093, correspond to vibration in an in-plane eigenmode and an out-of-plane eigenmode of the statically deformed beam, respectively. Because the beam is undamped the peaks should approach infinity.

In Fig. 4 the dimensionless cross-stiffness elements pertaining to the rotations $\phi_X(0)$ and $\phi_Z(0)$ of the root about the global X and Z axes are plotted vs the dimensionless frequency $\omega\sqrt{(mL^4/EI_\eta)}$ (when the root is released from the wall and when the tip still can move freely with the large static load P_{0Z}). The rotations $\phi_X(0)$ and $\phi_Z(0)$ are referred to degrees of freedom 4 and 6, respectively. The stiffness elements are calculated for both a nonfollower static moment and a semifollower static moment at the root. The stiffness elements $E_{46\,\text{is}}$ and $E_{64\,\text{is}}$ for the case with a semifollower static moment are equal, whereas the stiffness elements E_{461} and E_{641} for the case with a nonfollower large static moment are different. All curves approach infinity at the dimensionless resonance frequency

5.093. Each stiffness element passes zero once in the interval of the dimensionless frequency investigated. The element in row 1 and column 3 and the element in row 3 and column 1 of the matrix in Eqs. (40) correspond to the stiffness elements E_{46} and E_{64}, respectively, for zero frequency.

IX. Concluding Remarks

It has been theoretically shown how small linear displacements of a beam around an equilibrium configuration, as obtained from a nonlinear static analysis, should be treated. The intermediate state, i.e., the static equilibrium configuration, differs much from the initially straight state since the beam is subjected to large loads. Depending on the choice of the small moments, one can obtain symmetric or nonsymmetric stiffness and flexibility matrices. First, when the static moment from the nonlinear analysis is nonzero at one of the two boundaries, the stiffness matrix will be symmetric if the total sectional moment at the boundaries is taken as composed of one semifollower large moment and one small moment. The semifollower moment in a specific cross-section follows half the rotation of the beam lamina at that point. The stiffness matrix is the linear relation between the small translations and rotations of the ends, and the small applied forces and moments at the ends. Here, the latter are the difference between the actual moments and the semifollower moments. Second, if the total moment is composed of one large nonfollower moment and one small moment, then the stiffness matrix will be nonsymmetric. It is recalled that a real symmetric dynamic stiffness matrix corresponding to a symplectic transfer matrix represents a conservative nongyroscopic system for a harmonic excitation. No weak formulation is used, and no base functions are employed. The differential equations for both the static nonlinear case and the superimposed linear case are derived and solved numerically. This means that the beam is modeled as a continuum and that spatial discretization errors are avoided.

Appendix A: Equations Expressed in Local Components

The equilibrium equations (8) for the static case can be expressed in local components (see also Ref. 1) as

$$\frac{\partial \boldsymbol{F}_{0x}(s)}{\partial s} + \hat{\boldsymbol{\kappa}}_{0x}(s)\boldsymbol{F}_{0x}(s) + \boldsymbol{U}_{0x}(s) = \boldsymbol{0} \tag{A1a}$$

$$\frac{\partial \boldsymbol{M}_{0x}(s)}{\partial s} + \hat{\boldsymbol{\kappa}}_{0x}(s)\boldsymbol{M}_{0x}(s) + [\hat{\boldsymbol{e}}_x]_x\boldsymbol{F}_{0x}(s) + \boldsymbol{\Phi}_{0x}(s) = \boldsymbol{0} \tag{A1b}$$

Also the equilibrium equations (10) for the linear dynamic case can be expressed in local components as

$$\frac{\partial \boldsymbol{F}_{1x}(s,t)}{\partial s} + \hat{\boldsymbol{\kappa}}_{0x}(s)\boldsymbol{F}_{1x}(s,t) + \boldsymbol{U}_{1x}(s,t) = \boldsymbol{0} \tag{A2a}$$

$$\frac{\partial \boldsymbol{M}_{1x}(s,t)}{\partial s} + \hat{\boldsymbol{\kappa}}_{0x}(s)\boldsymbol{M}_{1x}(s,t) + [\hat{\boldsymbol{e}}_x]_x\boldsymbol{F}_{1x}(s,t) - \hat{\boldsymbol{F}}_{0x}(s)\frac{\partial \boldsymbol{u}_x(s,t)}{\partial s} - \hat{\boldsymbol{F}}_{0x}(s)\hat{\boldsymbol{\kappa}}_{0x}(s)\boldsymbol{u}_x(s,t) + \boldsymbol{\Phi}_{1x}(s,t) = \boldsymbol{0} \tag{A2b}$$

The kinematic relation between the derivative of local components $\boldsymbol{u}_x(s,t)$ of the dynamic displacement and the local components $\boldsymbol{\phi}_x(s,t)$ of the rotation of the beam cross section [see Eq. (13)] can be written

$$\frac{\partial \boldsymbol{u}_x(s,t)}{\partial s} = -\hat{\boldsymbol{\kappa}}_{0x}(s)\boldsymbol{u}_x(s,t) - [\hat{\boldsymbol{e}}_x]_x\boldsymbol{\phi}_x(s,t) \tag{A3}$$

The equilibrium equation (22) in the infinitesimally symplectic formulation, in local coordinates, is

$$\frac{\partial \boldsymbol{M}_{\mathrm{is}\,x}(s,t)}{\partial s} - \frac{1}{2}\hat{\boldsymbol{M}}_{0x}(s)\frac{\partial \boldsymbol{\phi}_x(s,t)}{\partial s} + \frac{1}{2}\hat{\boldsymbol{\phi}}_x(s,t)\frac{\partial \boldsymbol{M}_{0x}(s,t)}{\partial s} + \hat{\boldsymbol{\kappa}}_{0x}(s)\boldsymbol{M}_{\mathrm{is}\,x}(s,t) - \frac{1}{2}\hat{\boldsymbol{\kappa}}_{0x}(s)\hat{\boldsymbol{M}}_{0x}(s)\boldsymbol{\phi}_x(s,t) + [\hat{\boldsymbol{e}}_x]_x\boldsymbol{F}_{1x}(s,t) - \hat{\boldsymbol{F}}_{0x}(s)\frac{\partial \boldsymbol{u}_x(s,t)}{\partial s} - \hat{\boldsymbol{F}}_{0x}(s)\hat{\boldsymbol{\kappa}}_{0x}(s)\boldsymbol{u}_x(s,t) + \frac{1}{2}\hat{\boldsymbol{\phi}}_x(s,t)\boldsymbol{\Phi}_{0x}(s) + \boldsymbol{\Phi}_{\mathrm{is}\,x}(s,t) = \boldsymbol{0} \tag{A4}$$

The infinitesimally symplectic system, corresponding to Eq. (19), given in local coordinates can be written

$$\frac{\partial}{\partial s}\left[\boldsymbol{u}_x^T \quad \boldsymbol{\phi}_x^T \quad \boldsymbol{F}_{1x}^T \quad \boldsymbol{M}_{\mathrm{is}\,x}^T\right]^T(s,t) = \boldsymbol{A}_{\mathrm{is}\,x}(s)\left[\boldsymbol{u}_x^T \quad \boldsymbol{\phi}_x^T \quad \boldsymbol{F}_{1x}^T \quad \boldsymbol{M}_{\mathrm{is}\,x}^T\right]^T(s,t) + \boldsymbol{B}_x(s,t) \tag{A5}$$

The matrices $\boldsymbol{A}_{\mathrm{is}\,x}(s)$ and $\boldsymbol{B}_x(s,t)$ can be determined from Eqs. (23), (A1b), (A2a), (A3), and (A4):

$$\boldsymbol{A}_{\mathrm{is}\,x}(s) = \begin{bmatrix} -\hat{\boldsymbol{\kappa}}_{0x} & -[\hat{\boldsymbol{e}}_x]_x & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & -\hat{\boldsymbol{\kappa}}_{0x} + \frac{1}{2}(\boldsymbol{EI}_\xi)^{-1}\hat{\boldsymbol{M}}_{0x} & \boldsymbol{0} & (\boldsymbol{EI}_\xi)^{-1} \\ \boldsymbol{0} & \boldsymbol{0} & -\hat{\boldsymbol{\kappa}}_{0x} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_{\mathrm{is}\,x42} & -[\hat{\boldsymbol{e}}_x]_x & -\hat{\boldsymbol{\kappa}}_{0x} + \frac{1}{2}\hat{\boldsymbol{M}}_{0x}(\boldsymbol{EI}_\xi)^{-1} \end{bmatrix} \tag{A6a}$$

with

$$\boldsymbol{A}_{\mathrm{is}\,x42}(s) = \tfrac{1}{4}\hat{\boldsymbol{M}}_{0x}(\boldsymbol{EI}_\xi)^{-1}\hat{\boldsymbol{M}}_{0x} - \tfrac{1}{2}(\hat{\boldsymbol{F}}_{0x}[\hat{\boldsymbol{e}}_x]_x + [\hat{\boldsymbol{e}}_x]_x\hat{\boldsymbol{F}}_{0x}) \tag{A6b}$$

and

$$\boldsymbol{B}_x(s,t) = \left[\boldsymbol{0}^T \quad \boldsymbol{0}^T \quad -\boldsymbol{U}_{1x}^T \quad -\boldsymbol{\Phi}_{\mathrm{is}\,x}^T\right]^{\mathrm{T}} \tag{A7}$$

The matrix corresponding to the one given in Eqs. (A6) for the nonfollower sectional moment [see Eqs. (17b), (A2), and (A3)] is

$$\boldsymbol{A}_x(s) = \begin{bmatrix} -\hat{\boldsymbol{\kappa}}_{0x} & -[\hat{\boldsymbol{e}}_x]_x & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & -\hat{\boldsymbol{\kappa}}_{0x} + (\boldsymbol{EI}_\xi)^{-1}\hat{\boldsymbol{M}}_{0x} & \boldsymbol{0} & (\boldsymbol{EI}_\xi)^{-1} \\ \boldsymbol{0} & \boldsymbol{0} & -\hat{\boldsymbol{\kappa}}_{0x} & \boldsymbol{0} \\ \boldsymbol{0} & \frac{1}{2}\hat{\boldsymbol{\Phi}}_{0x} - \hat{\boldsymbol{F}}_{0x}[\hat{\boldsymbol{e}}_x]_x & -[\hat{\boldsymbol{e}}_x]_x & -\hat{\boldsymbol{\kappa}}_{0x} \end{bmatrix} \tag{A8}$$

Appendix B: Symplectic Matrices

According to Müller,[16] a symplectic matrix $\boldsymbol{T}$ is such that (here, matrices are bold face)

$$\boldsymbol{T}^T\boldsymbol{J}\boldsymbol{T} = c\boldsymbol{J}, \qquad c \neq 0 \tag{B1}$$

with $\boldsymbol{J}$ being the unit symplectic matrix defined by

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{0} \end{bmatrix} \tag{B2}$$

Here, the matrix $\boldsymbol{I}$ is the unit matrix. Let $\boldsymbol{T}$ be a function of the variable s, i.e., $\boldsymbol{T} = \boldsymbol{T}(s)$, and differentiate Eq. (B1) with respect to the variable s:

$$\boldsymbol{J}\left(\frac{\partial \boldsymbol{T}}{\partial s}\boldsymbol{T}^{-1}\right) = \left(\frac{\partial \boldsymbol{T}}{\partial s}\boldsymbol{T}^{-1}\right)^T \boldsymbol{J}^T \tag{B3}$$

Equation (B3) is the definition of an infinitesimally symplectic matrix $\boldsymbol{S}$

$$\boldsymbol{S} = \frac{\partial \boldsymbol{T}}{\partial s}\boldsymbol{T}^{-1} \tag{B4}$$

The definition in Eq. (B3) for the infinitesimally symplectic matrix $\boldsymbol{S}$ can also be written

$$\boldsymbol{JS} = (\boldsymbol{JS})^T \tag{B5}$$

If one partitions the infinitesimally symplectic matrix $\boldsymbol{S}$ into blocks as

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix} \tag{B6}$$

the definition in Eq. (B5) gives

$$\boldsymbol{b} = \boldsymbol{b}^T, \qquad \boldsymbol{c} = \boldsymbol{c}^T, \qquad \boldsymbol{d} = -\boldsymbol{a}^T \tag{B7}$$

Assume that a system of differential equations can be written in first-order form as

$$\frac{\mathrm{d}\boldsymbol{x}(s)}{\mathrm{d}s} = \boldsymbol{S}(s)\boldsymbol{x}(s) \tag{B8}$$

This system can be integrated from $s = 0$ to $s = L$ with a set of boundary conditions $\boldsymbol{x}(s = 0) = \boldsymbol{I}$. The result will be a relation between the variables $\boldsymbol{x}(s = 0)$ and $\boldsymbol{x}(s = L)$:

$$\boldsymbol{x}(s = L) = \boldsymbol{Tx}(s = 0) \tag{B9}$$

Here, the transfer matrix $\boldsymbol{T}$ will be symplectic if the the matrix $\boldsymbol{S}(s)$ is infinitesimally symplectic. A symplectic matrix $\boldsymbol{T}$ corresponds to a symmetric stiffness matrix with the first six variables in the column matrix $\boldsymbol{x}(s)$ being the displacements and the other six being the associated generalized forces.

Appendix C: Notation

Matrices and vectors are in bold face. A true vector $\boldsymbol{a}$ has no subscripts X, x, or ξ. When the vector $\boldsymbol{a}$ is projected in the xyz coordinate system, the representing column matrix is called $\boldsymbol{a}_x$. However, there are exceptions; the following variables are matrices: $\boldsymbol{e}, \boldsymbol{p}, \boldsymbol{E}, \boldsymbol{I}$, $\boldsymbol{J}$, and $\boldsymbol{E}$. The unit vectors $\boldsymbol{e}_\xi$, $\boldsymbol{e}_x$, etc., are true vectors. The components of the vector $\boldsymbol{e}_\xi$ taken in the xyz system are called $[\boldsymbol{e}_\xi]_x$. The index 0 refers to the basic static nonlinear problem and the index 1 refers to the additional dynamic linear problem. As a special case the additional linear problem can also be static. Subscript "is" indicates infinitesimally symplectic. The matrix $\hat{\boldsymbol{a}}_x$ is skew symmetric and built up of the elements of the column matrix $\boldsymbol{a}_x$:

$$\hat{\boldsymbol{a}}_x = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \tag{C1}$$

with

$$\boldsymbol{a}_x^T = [a_x \quad a_y \quad a_z] \tag{C2}$$

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